

A NOTE ON POLYLOGARITHMS AND INCOMPLETE GAMMA FUNCTION

*Dedicated to Academician Professor Gradimir Milovanović
on the occasion of his 70th birthday.*

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In this paper, we firstly introduce the polylogarithms and incomplete gamma function. Then, we claim that there is a relation between polylogarithms and a generalization of incomplete gamma function. Secondly, we give a formula related to polylogarithms. Also, we obtain a relation between incomplete gamma function and the derivatives of polylogarithms. Finally, we find a generating function for the values of incomplete gamma function.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Throughout this article, we use the following standard notations:

\mathbb{N} denotes the set of natural numbers, \mathbb{R} denotes the set of real numbers and \mathbb{C} denotes the set of complex numbers. Also,

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}$$

and the n -th derivative of any function f at z_0 is denoted by $f^{(n)}(z_0)$.

The polylogarithm (or de Jonquiére's function) $Li_s(z)$ (cf. [9]) is defined by

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z, s, 1)$$

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($s \in \mathbb{C}$ when $|z| < 1$; $Re(s) > 1$ when $|z| = 1$).

where $\Phi(z, s, w)$ is the Hurwitz-Lerch zeta function (cf. [2], [5]) defined by

$$\Phi(z, s, w) = \sum_{n=0}^{\infty} \frac{z^n}{(n+w)^s}$$

($s \in \mathbb{C}$ when $|z| < 1$; $Re(s) > 1$ when $|z| = 1$)

for $w \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$.

The integral representation of Hurwitz-Lerch zeta function is as follows (cf. [5]):

$$(1.1) \quad \Phi(z, s, w) = \frac{(-1)^{s-1}}{\Gamma(s)} \int_0^1 \frac{(\log t)^{s-1} t^{w-1}}{1-zt} dt$$

or

$$(1.2) \quad \Phi(z, s, w) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-wt}}{1-ze^{-t}} dt.$$

Let $N \in \mathbb{N}_0$. For $w = 1$ and $s = N + 1$, by using equation (1.1), we obtain

$$(1.3) \quad \begin{aligned} Li_{N+1}(z) &= z\Phi(z, N+1, 1) \\ &= z \frac{(-1)^N}{N!} \int_0^1 \frac{(\log t)^N}{1-zt} dt. \end{aligned}$$

By choosing $z = 1$ in (1.3), the polylogarithms can be reduced to Riemann zeta function $\zeta(N+1)$ (cf. [4]) given by

$$\zeta(N+1) = \sum_{n=1}^{\infty} \frac{1}{n^{N+1}}$$

where $N > 0$.

Recently, many authors have investigated the poly-Bernoulli polynomials $B_n^{(k)}(x)$ (cf. [6], [7], [8]), by using the properties of polylogarithms. A. Bayad and Y. Hamahata gave a generating function of $B_n^{(k)}(x)$ as follows (cf. [1]):

$$\frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}$$

for every integer k .

Also, Kaneko defined $B_n^{(k)}$ for every integer k by generating function (cf. [12])

$$\frac{Li_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}$$

where $B_n^{(k)} = B_n^{(k)}(0)$. In [12], an explicit formula for $B_n^{(k)}$ was given by

$$B_n^{(k)} = (-1)^n \sum_{m=0}^n \frac{(-1)^m m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\}}{(m+1)^k}$$

where $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$ is the Stirling number of the second kind (cf. [13]) defined by

$$\frac{1}{m!} (e^t - 1)^m = \sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \frac{t^n}{n!}.$$

Polylogarithms also appear in generating function of generalized harmonic numbers $H_{n,r}$ (cf. [10]) defined by

$$H_{n,r} = \sum_{k=1}^n \frac{1}{k^r}$$

where $r \in \mathbb{C}$.

From [1] and [12], we note that the polylogarithms are associated with poly-Bernoulli polynomials, specially poly-Bernoulli numbers and the Stirling number of the second kind.

Also, we introduce the incomplete gamma function $\Gamma(s, x)$ (cf. [10]):

$$(1.4) \quad \Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt$$

where $Re(s) > 0$ and $x \in \mathbb{R}$.

By choosing $x = 0$ in (1.4), we obtain the classical Euler gamma function (cf. [10]) given by

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt.$$

Polylogarithms and incomplete gamma function are useful functions in Analytic Number Theory and Mathematical Physics.

2. A GENERALIZATION OF INCOMPLETE GAMMA FUNCTION

In this section we introduce a generalization of incomplete gamma function $\Gamma_{\mu,z}(s, x)$. Then, by using the integral representation of polylogarithms $Li_{N+1}(z)$, we obtain a relation between polylogarithms and a general case of incomplete gamma function.

Recently, some authors have studied on generalizations of incomplete gamma function $\Gamma(\alpha, x; \beta)$ for $Re(\alpha) > 0$ and $\beta \in \mathbb{C}$.

In [11], Chaudhry and Zubair studied on the generalized incomplete gamma function given by the following integral:

$$(1.5) \quad \Gamma(\alpha, x; \beta) = \int_x^{\infty} t^{\alpha-1} e^{-t-\beta t^{-1}} dt.$$

In [3], Miller derived several reduction formulas for specializations of a certain generalized incomplete gamma function $\Gamma(\alpha, x; \beta)$ and its associated Kampé De Fériet function.

It is possible to define a generalization of different type from equation (1.5). Let $\mu \in \mathbb{R}$. Then, we define

$$\Gamma_{\mu,z}(N+1, x) = \int_x^{\infty} \frac{t^N e^{-\mu t}}{1 - ze^{-\mu t}} dt.$$

In this section, we claim that $\Gamma_{\mu,z}(s, x)$ is associated with polylogarithms. Firstly, we give the following key theorem for our claim:

Theorem 1. *Let $b > a > 0$ and $N \in \mathbb{N}_0$. Then, we have*

$$\int_{\log a}^{\log b} \frac{t^N e^t}{1 - ze^t} dt = \frac{1}{z} \sum_{k=0}^N \binom{N}{k} (-1)^k k! \{(\log b)^{N-k} Li_{k+1}(bz) - (\log a)^{N-k} Li_{k+1}(az)\}.$$

Proof. From equation (1.3), we get

$$\int_0^1 \frac{(\log t)^N}{1 - zt} dt = N! (-1)^N \frac{Li_{N+1}(z)}{z}.$$

We have

$$(1.6) \quad \int_a^b \frac{(\log t)^N}{1 - zt} dt = \int_0^{b-a} \frac{(\log(u+a))^N}{1 - za - zu} du,$$

by substituting $t = u + a$.

Then we have

$$(1.7) \quad \int_0^{b-a} \frac{(\log(u+a))^N}{1-za-zu} dt = \int_0^1 \frac{(\log((b-a)v+a))^N}{1-za-z(b-a)v} (b-a)dv,$$

by substituting $u = (b-a)v$ into the right side of equation (1.7).

From (1.6) and (1.7), we arrive

$$(1.8) \quad \int_a^b \frac{(\log t)^N}{1-zt} dt = \int_0^1 \frac{(\log((b-a)v+a))^N}{1-za-z(b-a)v} (b-a)dv.$$

On the other hand,

$$\int_0^b \frac{(\log t)^N}{1-zt} dt = \int_0^1 \frac{(\log b + \log u)^N}{1-bzu} bdu,$$

by substituting $t = bu$.

By using binomial expansion,

$$\begin{aligned} \int_0^b \frac{(\log t)^N}{1-zt} dt &= \int_0^1 \frac{(\log b + \log u)^N}{1-bzu} bdu \\ &= b \int_0^1 \frac{1}{1-bzu} \left\{ \sum_{k=0}^N \binom{N}{k} (\log b)^{N-k} (\log u)^k \right\} du \\ &= b \sum_{k=0}^N \binom{N}{k} (\log b)^{N-k} \left\{ \int_0^1 \frac{(\log u)^k}{1-bzu} du \right\} \\ &= b \sum_{k=0}^N \binom{N}{k} (\log b)^{N-k} k! (-1)^k \frac{Li_{k+1}(bz)}{bz} \\ (1.9) \quad &= \frac{1}{z} \sum_{k=0}^N \binom{N}{k} (\log b)^{N-k} k! (-1)^k Li_{k+1}(bz). \end{aligned}$$

Similarly,

$$(1.10) \quad \int_0^a \frac{(\log t)^N}{1-zt} dt = \frac{1}{z} \sum_{k=0}^N \binom{N}{k} (\log a)^{N-k} k! (-1)^k Li_{k+1}(az).$$

Suppose that $b > a > 0$. Then, we set

$$\int_a^b \frac{(\log t)^N}{1-zt} dt = \int_0^b \frac{(\log t)^N}{1-zt} dt - \int_0^a \frac{(\log t)^N}{1-zt} dt.$$

From (1.9) and (1.10), we have

$$(1.11) \quad \int_a^b \frac{(\log t)^N}{1-zt} dt = \frac{1}{z} \sum_{k=0}^N \binom{N}{k} k! (-1)^k \{ (\log b)^{N-k} Li_{k+1}(bz) - (\log a)^{N-k} Li_{k+1}(az) \}.$$

By using (1.8) and (1.11), we obtain

$$\begin{aligned} & \int_0^1 \frac{(\log((b-a)v+a))^N}{1-za-z(b-a)v} dv \\ &= \frac{1}{(b-a)z} \sum_{k=0}^N \binom{N}{k} k! (-1)^k \{ (\log b)^{N-k} Li_{k+1}(bz) - (\log a)^{N-k} Li_{k+1}(az) \}. \end{aligned}$$

By substituting $\log((b-a)v+a) = t$ into the left side of the above equation, we get

$$(1.12) \quad \int_0^1 \frac{(\log((b-a)v+a))^N}{1-za-z(b-a)v} dv = \int_{\log a}^{\log b} \frac{t^N}{1-za-z(e^t-a)} \frac{e^t}{(b-a)} dt.$$

By using the equation (1.12), we arrive at the desired result. \square

Remark 1. By choosing $a = 1$ in Theorem 1, we have

$$(1.13) \quad \int_0^{\log b} \frac{t^N e^t}{1-ze^t} dt = \frac{1}{z} \left\{ -Li_{N+1}(z) (-1)^N N! + \sum_{k=0}^N \binom{N}{k} (-1)^k k! (\log b)^{N-k} Li_{k+1}(bz) \right\}.$$

By substituting $t = -u$ into the left side of equation (1.13), we have

$$(1.14) \quad \int_0^{\log b} \frac{t^N e^t}{1-ze^t} dt = (-1)^{N+1} \int_0^{-\log b} \frac{u^N e^{-u}}{1-ze^{-u}} du = (-1)^N \int_{-\log b}^0 \frac{u^N e^{-u}}{1-ze^{-u}} du.$$

For $b > 1$,

$$(1.15) \quad \int_{-\log b}^{\infty} \frac{u^N e^{-u}}{1-ze^{-u}} du = \int_{-\log b}^0 \frac{u^N e^{-u}}{1-ze^{-u}} du + \int_0^{\infty} \frac{u^N e^{-u}}{1-ze^{-u}} du$$

where

$$(1.16) \quad \int_0^{\infty} \frac{u^N e^{-u}}{1-ze^{-u}} du = N! \Phi(z, N+1, 1) = N! \frac{Li_{N+1}(z)}{z}.$$

for $s = N + 1$ and $w = 1$ in equation (1.2).

From (1.14), (1.15) and (1.16), we have

$$\int_{-\log b}^{\infty} \frac{t^N e^{-t}}{1 - ze^{-t}} dt = \left\{ (-1)^N \int_0^{\log b} \frac{t^N e^t}{1 - ze^t} dt \right\} + N! \frac{Li_{N+1}(z)}{z}$$

or

$$\int_{-\log b}^{\infty} \frac{t^N e^{-t}}{1 - ze^{-t}} dt = \frac{(-1)^N}{z} \sum_{k=0}^N \binom{N}{k} (-1)^k k! (\log b)^{N-k} Li_{k+1}(bz)$$

by using equation (1.13).

For $b \rightarrow e^{-b}$, we get the following equation:

$$(1.17) \quad \int_b^{\infty} \frac{t^N e^{-t}}{1 - ze^{-t}} dt = \frac{N!}{z} \sum_{k=0}^N \frac{b^{N-k}}{(N-k)!} Li_{k+1}(e^{-b}z).$$

where $b \in \mathbb{R}$.

Remark 2. While $z \rightarrow 0$ in equation (1.17), we arrive

$$\Gamma(N+1, b) = \int_b^{\infty} t^N e^{-t} dt = e^{-b} N! \sum_{k=0}^N \frac{b^k}{k!}$$

where

$$\lim_{z \rightarrow 0} \frac{Li_{k+1}(e^{-b}z)}{z} = e^{-b}.$$

By substituting $b = \mu c$ into equation (1.17), we get

$$(1.18) \quad \int_{\mu c}^{\infty} \frac{t^N e^{-t}}{1 - ze^{-t}} dt = \frac{N!}{z} \sum_{k=0}^N \frac{(\mu c)^{N-k}}{(N-k)!} Li_{k+1}(e^{-\mu c}z).$$

By substituting $t = \mu v$ into the left side of equation (1.18), we arrive at the following corollary:

Corollary 1. Let $c, \mu \in \mathbb{R}$ and $N \in \mathbb{N}_0$. Then, we have

$$\Gamma_{\mu,z}(N+1, c) = \int_c^{\infty} \frac{v^N e^{-\mu v}}{1 - ze^{-\mu v}} dv = \frac{N!}{\mu^{N+1}z} \sum_{k=0}^N \frac{(\mu c)^{N-k}}{(N-k)!} Li_{k+1}(e^{-\mu c}z).$$

Remark 3. By choosing $\mu = 1$ in Corollary 1, we have

$$\lim_{z \rightarrow 0} \Gamma_{1,z}(N+1, c) = \Gamma(N+1, c).$$

Therefore, we note that $\Gamma_{\mu,z}(N+1, c)$ is a generalization of the incomplete gamma function $\Gamma(N+1, c)$.

3. MAIN RESULTS

In this section, we firstly give a formula for the polylogarithms. Secondly, we obtain a relation between incomplete gamma function and the derivatives of polylogarithms. Finally, we find the generating function for the values of incomplete gamma function.

From equation (1.3), we get

$$(1.19) \quad \int_0^1 \frac{(\log t)^N}{1 - Mzt} dt = \frac{Li_{N+1}(Mz)}{Mz} (-1)^N N!$$

By substituting $u = Mt$ into the left side of equation (1.19), we get

$$\int_0^1 \frac{(\log t)^N}{1 - Mzt} dt = \frac{1}{M} \int_0^M \frac{(\log u - \log M)^N}{1 - zu} du$$

Denote such a partition P , that is $P = \{0, 1, 2, \dots, M\}$. For $k \in \mathbb{N}_0$, we integrate on $[k, k+1]$:

$$(1.20) \quad \int_0^M \frac{(\log u - \log M)^N}{1 - zu} du = \sum_{k=0}^{M-1} \int_k^{k+1} \frac{(\log u - \log M)^N}{1 - zu} du.$$

By substituting $u = v+k$ into the left side of equation (1.20) for each integral, we have

$$\int_k^{k+1} \frac{(\log u - \log M)^N}{1 - zu} du = \int_0^1 \frac{(\log(v+k) - \log M)^N}{1 - kz - zv} dv.$$

Then, we arrive

$$\int_0^1 \frac{(\log t)^N}{1 - Mzt} dt = \frac{1}{M} \sum_{k=0}^{M-1} \int_0^1 \frac{(\log(v+k) - \log M)^N}{1 - kz - zv} dv.$$

By using binomial expansion,

$$(1.21) \quad \int_0^1 \frac{(\log(v+k) - \log M)^N}{1 - kz - zv} dv = \int_0^1 \frac{1}{1 - kz - zv} \left\{ \sum_{j=0}^N \binom{N}{j} (-\log M)^{N-j} (\log(v+k))^j \right\} dv$$

$$(1.22) \quad = \sum_{j=0}^N \binom{N}{j} (-\log M)^{N-j} \left\{ \int_0^1 \frac{(\log(v+k))^j}{1 - kz - zv} dv \right\}$$

Also, by substituting $\log(v+k) = t$ into the right side of equation (1.22), we have

$$\int_0^1 \frac{(\log(v+k))^j}{1-kz-zv} dv = \int_{\log k}^{\log(k+1)} \frac{t^j e^t}{1-ze^t} dt.$$

Hence,

$$\begin{aligned} \frac{Li_{N+1}(Mz)}{Mz} (-1)^N N! &= \int_0^1 \frac{(\log t)^N}{1-Mzt} dt \\ &= \frac{1}{M} \sum_{k=0}^{M-1} \sum_{j=0}^N \binom{N}{j} (-\log M)^{N-j} \left\{ \int_{\log k}^{\log(k+1)} \frac{t^j e^t}{1-ze^t} dt \right\} \\ &= \frac{1}{M} \sum_{j=0}^N \binom{N}{j} (-\log M)^{N-j} \left\{ \int_{-\infty}^0 \frac{t^j e^t}{1-ze^t} dt \right\} \\ (1.23) \quad &+ \frac{1}{M} \sum_{k=1}^{M-1} \sum_{j=0}^N \binom{N}{j} (-\log M)^{N-j} \left\{ \int_{\log k}^{\log(k+1)} \frac{t^j e^t}{1-ze^t} dt \right\}. \end{aligned}$$

where

$$\begin{aligned} \int_{-\infty}^0 \frac{t^j e^t}{1-ze^t} dt &= (-1)^j \int_0^{\infty} \frac{t^j e^{-t}}{1-ze^{-t}} dt \\ &= (-1)^j j! \Phi(z, j+1, 1) \\ &= (-1)^j j! \frac{Li_{j+1}(z)}{z}. \end{aligned}$$

By using Theorem 1 for $b = k+1$ and $a = k$, the equation (1.23) can be written as

$$\begin{aligned} \frac{Li_{N+1}(Mz)}{Mz} (-1)^N N! - \frac{1}{Mz} \sum_{j=0}^N \binom{N}{j} (-\log M)^{N-j} (-1)^j j! Li_{j+1}(z) &= \\ \frac{1}{Mz} \sum_{k=1}^{M-1} \sum_{j=0}^N \binom{N}{j} (-\log M)^{N-j} \sum_{r=0}^j \binom{j}{r} (-1)^r r! \left\{ \begin{array}{l} (\log(k+1))^{j-r} Li_{r+1}(kz+z) \\ -(\log k)^{j-r} Li_{r+1}(kz) \end{array} \right\}. \end{aligned}$$

Consequently, we arrive at the following theorem:

Theorem 2. *Let $M, N \in \mathbb{N}_0$. Then, we have*

$$\begin{aligned} (-1)^N N! Li_{N+1}((M+2)z) - \sum_{j=0}^N \binom{N}{j} (-\log(M+2))^{N-j} (-1)^j j! Li_{j+1}(z) &= \\ \sum_{k=0}^M \sum_{j=0}^N \sum_{r=0}^j \binom{N}{j} \binom{j}{r} (-\log(M+2))^{N-j} (-1)^r r! \left\{ \begin{array}{l} (\log(k+2))^{j-r} Li_{r+1}((k+2)z) \\ -(\log(k+1))^{j-r} Li_{r+1}((k+1)z) \end{array} \right\}. \end{aligned}$$

For our second main result, we use

$$(1.24) \quad \frac{e^{-\mu t}}{1 - ze^{-\mu t}} = \frac{1}{z} \sum_{n=1}^{\infty} z^n e^{-n\mu t}.$$

From equation (1.24), we have

$$(1.25) \quad \begin{aligned} \int_c^{\infty} \frac{t^N ze^{-\mu t}}{1 - ze^{-\mu t}} dt &= \int_c^{\infty} t^N \left(\sum_{n=1}^{\infty} z^n e^{-n\mu t} \right) dt \\ &= \sum_{n=1}^{\infty} z^n \left(\int_c^{\infty} t^N e^{-n\mu t} dt \right). \end{aligned}$$

By substituting $n\mu t = u$, we get

$$(1.26) \quad \int_c^{\infty} t^N e^{-n\mu t} dt = \frac{1}{(n\mu)^{N+1}} \int_{n\mu c}^{\infty} u^N e^{-u} du.$$

From equation (1.26), equation (1.25) becomes

$$\begin{aligned} \int_c^{\infty} \frac{t^N ze^{-\mu t}}{1 - ze^{-\mu t}} dt &= \frac{1}{\mu^{N+1}} \sum_{n=1}^{\infty} \frac{z^n}{n^{N+1}} \left(\int_{n\mu c}^{\infty} u^N e^{-u} du \right) \\ &= \frac{1}{\mu^{N+1}} \sum_{n=1}^{\infty} \frac{\Gamma(N+1, n\mu c)}{n^{N+1}} z^n \end{aligned}$$

Then, from Corollary 1, we deduce

$$\sum_{n=1}^{\infty} \frac{\Gamma(N+1, n\mu c)}{n^{N+1}} z^n = N! \sum_{k=0}^N \frac{(\mu c)^{N-k}}{(N-k)!} Li_{k+1}(e^{-\mu c} z)$$

or

$$(1.27) \quad \sum_{n=1}^{\infty} \frac{\Gamma(N+1, n\xi)}{n^{N+1}} z^n = N! \sum_{k=0}^N \frac{\xi^{N-k}}{(N-k)!} Li_{k+1}(e^{-\xi} z)$$

for $\xi = \mu c$.

We define

$$f_{N,\xi}(z) = N! \sum_{k=0}^N \frac{\xi^{N-k}}{(N-k)!} Li_{k+1}(e^{-\xi} z).$$

The n -th derivative of function $f_{N,\xi}$:

$$\frac{d^n}{dz^n} f_{N,\xi}(z) = N! e^{-n\xi} \sum_{k=0}^N \frac{\xi^{N-k}}{(N-k)!} Li_{k+1}^{(n)}(e^{-\xi} z).$$

For $z = 0$,

$$f_{N,\xi}^{(n)}(0) = N!e^{-n\xi} \sum_{k=0}^N \frac{\xi^{N-k}}{(N-k)!} Li_{k+1}^{(n)}(0).$$

In the left side of equation (1.27), by using the property of Taylor series, we obtain

$$\begin{aligned} \frac{\Gamma(N+1, n\xi)}{n^{N+1}} &= \frac{f_{N,\xi}^{(n)}(0)}{n!} \\ &= \frac{N!}{n!} e^{-n\xi} \sum_{k=0}^N \frac{\xi^{N-k}}{(N-k)!} Li_{k+1}^{(n)}(0). \end{aligned}$$

Consequently, we arrive at the following theorem:

Theorem 3. *Let $\xi \in \mathbb{R}$, $N \in \mathbb{N}_0$ and $1 \leq n \in \mathbb{N}$. Then, we have*

$$(1.28) \quad \frac{e^{n\xi} n! \Gamma(N+1, n\xi)}{n^{N+1} N!} = \sum_{k=0}^N \frac{\xi^{N-k}}{(N-k)!} Li_{k+1}^{(n)}(0).$$

Remark 4. *The n -th derivative of function $Li_{k+1}(z)$ is given as follows:*

$$\begin{aligned} \frac{d^n}{dz^n} Li_{k+1}(z) &= \sum_{m=n}^{\infty} \frac{(m-1)(m-2)(m-3) \cdots (m-n+1)}{m^k} z^{m-n} \\ &= \sum_{m=0}^{\infty} \frac{(m+n-1)(m+n-2)(m+n-3) \cdots (m+1)}{(m+n)^k} z^m \\ &= \sum_{m=0}^{\infty} \frac{(m+n-1)!}{(m+n)^k m!} z^m. \end{aligned}$$

For $z = 0$, we arrive

$$(1.29) \quad Li_{k+1}^{(n)}(0) = \frac{(n-1)!}{n^k}.$$

From (1.28) and (1.29), we get

$$\frac{e^{n\xi} \Gamma(N+1, n\xi)}{n^N N!} = \sum_{k=0}^N \frac{\xi^{N-k}}{(N-k)! n^k}.$$

We expand the series both side of the above equation for $N \in \mathbb{N}_0$:

$$\begin{aligned} e^{n\xi} \sum_{N=0}^{\infty} \Gamma(N+1, n\xi) \frac{(t/n)^N}{N!} &= \sum_{N=0}^{\infty} \sum_{k=0}^N \frac{\xi^{N-k}}{(N-k)! n^k} t^N \\ &= \sum_{N=0}^{\infty} \sum_{k=0}^N \frac{(t\xi)^{N-k}}{(N-k)!} \left(\frac{t}{n}\right)^k. \end{aligned}$$

By using Cauchy product for the right side of the above equation, we have

$$\begin{aligned} e^{n\xi} \sum_{N=0}^{\infty} \Gamma(N+1, n\xi) \frac{(t/n)^N}{N!} &= \sum_{N=0}^{\infty} \sum_{k=0}^N \frac{(t\xi)^{N-k}}{(N-k)!} \left(\frac{t}{n}\right)^k \\ &= \left(\sum_{N=0}^{\infty} \frac{(t\xi)^N}{N!} \right) \left(\sum_{N=0}^{\infty} (t/n)^N \right) \\ &= e^{t\xi} \frac{1}{1 - \frac{t}{n}} \end{aligned}$$

where $|t/n| < 1$.

For $t \rightarrow nt$,

$$\sum_{N=0}^{\infty} \Gamma(N+1, n\xi) \frac{t^N}{N!} = \frac{e^{-n\xi(1-t)}}{1-t}$$

where $|t| < 1$.

Hence, we obtain the following corollary for $n\xi = x$:

Corollary 2. *Let $x \in \mathbb{R}$ and $|t| < 1$. Then, the values of incomplete gamma function is given by the following generating function:*

$$\frac{e^{-x(1-t)}}{1-t} = \sum_{N=0}^{\infty} \Gamma(N+1, x) \frac{t^N}{N!}.$$

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