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Partial differential equations for a new family of numbers and polynomials unifying the Apostol-type numbers and the Apostol-type polynomials

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A B S T R A C T

The main motivation of this paper is to investigate some derivative properties of the generating functions for the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$, which were recently introduced by Simsek [30]. We give functional equations and differential equations (PDEs) of these generating functions. By using these functional and differential equations, we derive not only recurrence relations, but also several other identities and relations for these numbers and polynomials. Our identities include the Apostol–Bernoulli numbers, the Apostol–Euler numbers, the Stirling numbers of the first kind, the Cauchy numbers and the Hurwitz–Lerch zeta functions. Moreover, we give hypergeometric function representation for an integral involving these numbers and polynomials. Finally, we give infinite series representations of the numbers $Y_n(\lambda)$, the

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Apostol–Bernoulli numbers and Apostol–Bernoulli polynomials
Apostol–Euler numbers and Apostol–Bernoulli polynomials
Daehee and Changhee numbers
Stirling numbers of the first kind
Cauchy numbers
Humbert polynomials
Lucas numbers
Binomial coefficients

Changhee numbers, the Daehee numbers, the Lucas numbers and the Humbert polynomials.

1. Introduction and preliminaries

In this paper, we are motivated to consider an interesting new family of polynomials $Y_n(x;\lambda)$ given by the following generating function:

$$F(t,x,\lambda)=\frac{2(1+\lambda t)^x}{\lambda(1+\lambda t)-1}=\sum_{n=0}^{\infty}Y_n(x;\lambda)\,\frac{t^n}{n!},\tag{1}$$

which were recently introduced and investigated by Simsek (see [30]) by using a new family of special numbers $Y_{n,\chi}(\lambda,q)$ attached to a Dirichlet character χ. When $q\to 1$ and $\chi\equiv 1$, the numbers $Y_{n,\chi}(\lambda,q)$ reduce to the following generating function for the numbers $Y_n(\lambda)$:

$$F(t,\lambda)=\frac{2}{\lambda(1+\lambda t)-1}=\sum_{n=0}^{\infty}Y_n(\lambda)\,\frac{t^n}{n!}.\tag{2}$$

We observe that (*cf.* [30])

$$Y_n(\lambda)=Y_n(0;\lambda).$$

By using the generating function in (1), a few values of the polynomials $Y_n(x;\lambda)$ are given as follows:

$$Y_0(x;\lambda)=\frac{2}{\lambda-1},\qquad Y_1(x;\lambda)=\frac{2\lambda}{\lambda-1}\,x-\frac{2\lambda^2}{(\lambda-1)^2},$$
$$Y_2(x;\lambda)=\frac{2\lambda^2}{\lambda-1}\,x^2-\frac{6\lambda^3-2\lambda^2}{(\lambda-1)^2}\,x+\frac{4\lambda^4}{(\lambda-1)^3},$$
$$Y_3(x;\lambda)=\frac{2\lambda^3}{\lambda-1}\,x^3-\frac{12\lambda^4-6\lambda^3}{(\lambda-1)^2}\,x^2+\frac{22\lambda^5-14\lambda^4+4\lambda^3}{(\lambda-1)^3}\,x-\frac{12\lambda^6}{(\lambda-1)^4},$$

and so on. Thus, clearly a few values of the numbers $Y_n(\lambda)$ are given by (*cf.* [30])

$$Y_0(\lambda) = \frac{2}{\lambda-1}, \qquad Y_1(\lambda) = -\frac{2\lambda^2}{(\lambda-1)^2}, \qquad Y_2(\lambda) = \frac{4\lambda^4}{(\lambda-1)^3},$$
$$Y_3(\lambda) = -\frac{12\lambda^6}{(\lambda-1)^4}, \qquad Y_4(\lambda) = \frac{48\lambda^8}{(\lambda-1)^5},$$

and so on.

As stated by Simsek [30], there exist some significant combinatorial identities which are essentially associated with the numbers $Y_n(\lambda)$, the polynomials $Y_n(x;\lambda)$ and several other special numbers and polynomials including the Apostol type numbers and polynomials, the Stirling numbers, the Cauchy numbers and the Bernoulli numbers of the second kind. Recently, by using differential equations derivable from the generating function of Changhee polynomials, Kim et al. [16] obtained several identities and formulas related to Changhee polynomials. By using similar method and technique, we give some PDEs for the generating functions for the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$ with respect to the variables x and t as well as the parameter λ. Moreover, by using some techniques involving the generating functions, functional equations, partial differential equations, we investigate the various fundamental properties of the generating functions for the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$. By using Apostol type numbers, the Stirling numbers of the first kind, the Cauchy numbers and the Hurwitz–Lerch zeta function, we also derive several identities and relations associated with the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$. In addition to the PDEs of the generating functions for these numbers and polynomials, we give integral expressions for both the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$ in terms of the Gauss hypergeometric function. Furthermore, we investigate infinite series representations of not only the ratios of the numbers $Y_n(\lambda)$, the Changhee numbers, the Daehee numbers to one another, but also inverses of the numbers $Y_n(\lambda)$. We give some formulas for these infinite series and we obtain relations between these infinite series and the Humbert polynomials and the Lucas numbers.

In our present investigation, we need the following definitions, notations, and other preliminaries. Throughout this paper, we write

$$\mathbb{N} = \{1, 2, 3, \cdots\} \qquad \text{and} \qquad \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

and denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the set of integers, the set of real numbers and the set of complex numbers, respectively. We also assume *tacitly* that (*cf.* [1–35,39]; see also the references cited therein)

$$0^n = \begin{cases} 1 & (n = 0) \\ 0 & (n \in \mathbb{N}). \end{cases}$$

Moreover, for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have

$$\binom{\lambda}{0} = 1 \qquad \text{and} \qquad \binom{\lambda}{n} = \frac{(\lambda)_n}{n!},$$

where $(\lambda)_n$ is the falling factorial defined by

$$(\lambda)_n = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1) \qquad (\lambda \in \mathbb{C};\ n \in \mathbb{N})$$

and

$$(\lambda)_0 = 1 \qquad (\lambda \in \mathbb{C}).$$

On the other hand, the rising factorial $(\lambda)^{(n)}$ is defined similarly by

$$(\lambda)^{(n)} = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1) \qquad (\lambda \in \mathbb{C};\ n \in \mathbb{N})$$

and

$$(\lambda)^{(0)} = 1 \qquad (\lambda \in \mathbb{C}).$$

Clearly, we have

$$(\lambda)^{(n)} = (-1)^n (-\lambda)_n \qquad (\lambda \in \mathbb{C};\ n \in \mathbb{N}_0). \tag{3}$$

The Daehee polynomials $D_n(x)$ defined by the following generating function (*cf.* [12, 29,31]):

$$F_D(x,t) = (1+t)^x \; F_D(t) = \sum_{n=0}^{\infty} D_n(x) \, \frac{t^n}{n!} \tag{4}$$

and the corresponding Daehee numbers $D_n = D_n(0)$ are given by

$$F_D(t) = F_D(0,t) = \frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \, \frac{t^n}{n!}. \tag{5}$$

By using (5), we get the following explicit formula for the Daehee numbers D_n (*cf.* [7, 12]):

$$D_n = \frac{(-1)^n n!}{n+1}.$$

The Peters polynomials $s_k(x;\lambda,\mu)$, which are a member of the family of the Sheffer polynomials, are given by the following generating functions (*cf.* [10,13,27]):

$$\frac{(1+t)^x}{\left[1+(1+t)^\lambda\right]^\mu} = \sum_{n=0}^{\infty} s_k(x;\lambda,\mu) \, \frac{t^n}{n!},$$

which, in the special case when $\lambda = \mu = 1$, reduces immediately to the Changhee polynomials $\mathrm{Ch}_n(x)$ defined by the following generating function (*cf.* [14,15]):

$$F_{\mathrm{Ch}}(x,t) = (1+t)^x \ F_{\mathrm{Ch}}(t) = \sum_{n=0}^{\infty} \mathrm{Ch}_n(x) \ \frac{t^n}{n!}, \tag{6}$$

which, for $x = 0$, yields the Changhee numbers Ch_n which is defined by means of the following generating function (*cf.* [14,15]). By using (7):

$$F_{\mathrm{Ch}}(t) = \frac{2}{t+2} = \sum_{n=0}^{\infty} \mathrm{Ch}_n \ \frac{t^n}{n!}. \tag{7}$$

The explicit formula for the Changhee numbers Ch_n is given by (*cf.* [15])

$$\mathrm{Ch}_n = \frac{(-1)^n \, n!}{2^n}.$$

Our results are also associated with many other well-known numbers and well-known polynomials, which are recalled below along with their generating functions.

The Apostol–Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ are defined by the following generating function (see [1,32]):

$$\frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda) \ \frac{t^n}{n!} \tag{8}$$

$$(\lambda \in \mathbb{C}; \ |t| < 2\pi \quad \text{when} \quad \lambda = 1; \ |t| < |\log \lambda| \quad \text{when} \quad \lambda \neq 1; \ ).$$

For $x = 0$, these polynomials are reduced to the Apostol–Bernoulli numbers $\mathcal{B}_n(\lambda)$ which are given by the following generating function (*cf.* [1,21,33,36,38]):

$$\frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(\lambda) \ \frac{t^n}{n!}. \tag{9}$$

Also, in their special case when $\lambda = 1$, the Apostol–Bernoulli numbers $\mathcal{B}_n(\lambda)$ are reduced to the classical Bernoulli numbers B_n (*cf.* [1–36]; see also the references cited therein):

$$B_n = \mathcal{B}_n(1).$$

By using the above generating functions for the Apostol–Bernoulli numbers $\mathcal{B}_n(\lambda)$ and the Apostol–Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ together with the method of umbral calculus convention, a few of these numbers and polynomials are computed as follows (*cf.* [1–36]; see also the related references cited therein):

$$\mathcal{B}_0(\lambda) = 0, \ \ \mathcal{B}_1(\lambda) = \frac{1}{\lambda - 1}, \ \ \mathcal{B}_2(\lambda) = -\frac{2\lambda}{(\lambda-1)^2}, \ \ \mathcal{B}_3(\lambda) = \frac{3\lambda(\lambda+1)}{(\lambda-1)^3}, \cdots$$

and

$$\mathcal{B}_0(x;\lambda)=0,\;\;\mathcal{B}_1(x;\lambda)=\frac{1}{\lambda-1},\;\;\mathcal{B}_2(x;\lambda)=\frac{1}{\lambda-1}\,x-\frac{2\lambda}{(\lambda-1)^2},$$
$$\mathcal{B}_3(x;\lambda)=\frac{3}{\lambda-1}\,x^2-\frac{6\lambda}{(\lambda-1)^2}\,x+\frac{3\lambda(\lambda+1)}{(\lambda-1)^3},\cdots.$$

The Apostol–Euler polynomials $\mathcal{E}_n(x;\lambda)$ are given by the following generating function (*cf.* [5,11,25,33,35,38]; see also the related references cited therein):

$$\frac{2e^{xt}}{\lambda e^t+1}=\sum_{n=0}^{\infty}\mathcal{E}_n(x;\lambda)\,\frac{t^n}{n!} \tag{10}$$
$$\left(\lambda\in\mathbb{C};\;|t|<\pi\quad\text{when}\quad\lambda=1;\;|t|<|\ln(-\lambda)|\quad\text{when}\quad\lambda\neq1\right).$$

In the special case when $x=0$, these polynomials are reduced to the Apostol–Euler numbers $\mathcal{E}_n(\lambda)$:

$$\mathcal{E}_n(\lambda)=\mathcal{E}_n(0;\lambda),$$

which are given by the following generating function:

$$\frac{2}{\lambda e^t+1}=\sum_{n=0}^{\infty}\mathcal{E}_n(\lambda)\,\frac{t^n}{n!}.$$

Moreover, for $\lambda=1$, the Apostol–Euler polynomials $\mathcal{E}_n(x;\lambda)$ are reduced to the classical Euler polynomials of the first kind (*cf.* [6–36]; see also the related references cited therein):

$$E_n(x)=\mathcal{E}_n(x;1). \tag{11}$$

The classical Euler numbers E_n of the first kind are defined by the following generating function:

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\,\frac{t^n}{n!}\qquad(|t|<\pi).$$

In particular, for $\lambda=1$, we have (*cf.* [6–36]; see also the related references cited therein)

$$E_n=\mathcal{E}_n(1). \tag{12}$$

By using the above generating functions for the Apostol–Euler polynomials $\mathcal{E}_n(x;\lambda)$ and the Apostol–Euler numbers $\mathcal{E}_n(\lambda)$ together with the method of umbral calculus convention, a few of these numbers are computed as follows (*cf.* [6–36]; see also the related references cited therein):

$$\mathcal{E}_0(\lambda)=\frac{2}{\lambda+1},\quad \mathcal{E}_1(\lambda)=-\frac{2\lambda}{(\lambda+1)^2},\quad \mathcal{E}_2(\lambda)=\frac{2\lambda(\lambda-1)}{(\lambda+1)^3},$$
$$\mathcal{E}_3(\lambda)=-\frac{2\lambda\left(\lambda^2-4\lambda+1\right)}{(\lambda+1)^4},\cdots.$$

For several finite sums involving the interpolation functions of the Bernoulli and Euler polynomials, and for several identities, relations and formulas associated with the classical Bernoulli and Euler polynomials, the reader should consult [32].

The Stirling numbers $S_1(n,k)$ of the first kind are defined by the following generating function:

$$(x)_n=\sum_{k=0}^{n} S_1(n,k)\,x^k \tag{13}$$

or, equivalently, by (*cf.* [2,26,28,35,36]; see also the references cited therein)

$$\sum_{n=k}^{\infty} S_1(n,k)\ \frac{t^n}{n!}=\frac{[\log(1+t)]^k}{k!}.$$

The Bernoulli numbers $b_n(0)$ of the *second* kind, which are also called the Cauchy numbers, are defined by means of the following generating function (*cf.* [27, p. 116]):

$$F_C(t)=\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty} b_n(0)\ \frac{t^n}{n!}. \tag{14}$$

The numbers $b_n(0)$ are calculated by the following formula (*cf.* [27, pp. 113–117]):

$$b_n(0)=\int_0^1 (x)_n\ \mathrm{d}x,$$

so that a few of the Cauchy numbers $b_n(0)$ are given by (*cf.* [17,26] and [27, pp. 113–117]; see also the references cited therein)

$$b_0(0)=1,\quad b_1(0)=\frac{1}{2},\quad b_2(0)=-\frac{1}{12},\quad b_3(0)=\frac{1}{24},\quad b_4(0)=-\frac{19}{720},\cdots.$$

The Humbert polynomials $\Pi_{n,m}^{(\lambda)}(x)$ were defined by Humbert [9] by means of the following generating function (*cf.* [9], [37, p. 86, Eq. 1.11 (26)], [24]):

$$(1-mxt+t^m)^{-\lambda}=\sum_{n=0}^{\infty}\Pi_{n,m}^{(\lambda)}(x)\,t^n.$$

The recurrence relation for these polynomials is given as follows (*cf.* [6,23]; see also the references cited therein):

$$(n+1)\,\Pi_{n+1,m}^{(\lambda)}(x) - mx\,(n+\lambda)\,\Pi_{n,m}^{(\lambda)}(x) - (n+m\lambda-m+1)\,\Pi_{n-m+1,m}^{(\lambda)}(x) = 0.$$

The generalized Humbert polynomials $P_n(m,x,y,p,C)$ are defined by the following generating function (*cf.* [6,8,23,24]; see also [37, p. 86, Eq. 1.11 (29)]):

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m,x,y,p,C)t^n,$$

so that, clearly, we have

$$P_n(m,x,1,-\lambda,1) = \Pi_{n,m}^{(\lambda)}(x)\,.$$

We summarize the presentation of our results as follows. In Section 2 and Section 3, by using some general techniques involving the partial differential equations of the function $F(t,x,\lambda)$ with respect to x, t and λ, we first derive several identities, functional equations and recurrence relations associated with the numbers $Y_n(\lambda)$, the polynomials $Y_n(x;\lambda)$, the Daehee numbers and polynomials, and the Changhee numbers and polynomials. In Section 3, we make use of these derivative formulas in order to derive some identities involving the Stirling numbers of the first kind and the Cauchy numbers. In Section 4, we give further identities and relations associated with the numbers $Y_n(\lambda)$ together with the Hurwitz–Lerch zeta function, the Apostol–Bernoulli numbers, and the Apostol–Euler numbers. Furthermore, we give some hypergeometric function representation for the integral of the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$. We also give some infinite series representation including the numbers $Y_n(\lambda)$, the Daehee numbers, the Changhee numbers, the Lucas numbers, and the Humbert polynomials. Finally, in Section 5, we present a number of concluding remarks and observations.

2. Partial derivatives of the generating function $F(t,x,\lambda)$

This section deals with new functional equations related to the polynomials $Y_n(x;\lambda)$ and the differential equations associated with their generating functions.

Differentiating both side of (1) with respect to t, we get the following partial differential equation:

$$\frac{\partial}{\partial t}\{F(t,x,\lambda)\} = F(t,x,\lambda)\left(\frac{\lambda x}{1+\lambda t} - \frac{\lambda^2}{2}F(t,\lambda)\right). \tag{15}$$

Combining (1) and (2) with the partial differential equation in (15), we find for $|\lambda t| < 1$ that

$$\sum_{n=0}^{\infty} Y_{n+1}(x;\lambda)\,\frac{t^n}{n!} = \lambda x \sum_{n=0}^{\infty} (-\lambda t)^n \sum_{n=0}^{\infty} Y_n(x;\lambda)\,\frac{t^n}{n!} - \frac{\lambda^2}{2} \sum_{n=0}^{\infty} Y_n(\lambda)\,\frac{t^n}{n!} \sum_{n=0}^{\infty} Y_n(x;\lambda)\,\frac{t^n}{n!}.$$

By using the Cauchy product in the above equation, we have

$$\sum_{n=0}^{\infty} Y_{n+1}(x;\lambda)\,\frac{t^n}{n!} = \lambda x \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-\lambda)^{n-k}}{k!}\, Y_k(x;\lambda) \right) t^n - \frac{\lambda^2}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} Y_k(x;\lambda)\, Y_{n-k}(\lambda)\,\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following result.

Theorem 1. *It is asserted that*

$$Y_{n+1}(x;\lambda) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} Y_k(x;\lambda) \left[2(-1)^{n-k} \lambda^{n-k+1} x (n-k)! - \lambda^2 Y_{n-k}(\lambda) \right].$$

Corollary 1. *The following assertion holds true*:

$$Y_{n+1}(x;\lambda) = \sum_{k=0}^{n} \binom{n}{k} Y_k(x;\lambda) \cdot \left[(-1)^{n-k} \lambda^{n-k+1} x (n-k)! - 2^{n-k} \left(\frac{\lambda^2}{\lambda-1} \right)^{n-k+1} \mathrm{Ch}_{n-k} \right].$$

Proof. By using (2), we define the following functional equation:

$$\frac{\lambda^2}{2} F(t,\lambda) = \frac{\lambda^2}{\lambda-1} F_{\mathrm{Ch}} \left(\frac{2\lambda^2}{\lambda-1} t \right). \tag{16}$$

Combining (16) with equation (15), we find for $|\lambda t| < 1$ that

$$\sum_{n=0}^{\infty} Y_{n+1}(x;\lambda)\,\frac{t^n}{n!} = \sum_{n=0}^{\infty} Y_n(x;\lambda)\,\frac{t^n}{n!} \cdot \left[\lambda x \sum_{n=0}^{\infty} (-\lambda t)^n - \frac{\lambda^2}{\lambda-1} \sum_{n=0}^{\infty} \left(\frac{2\lambda^2}{\lambda-1} \right)^n \mathrm{Ch}_n \frac{t^n}{n!} \right].$$

After some elementary calculations with the Cauchy product, we obtain

$$\sum_{n=0}^{\infty} Y_{n+1}(x;\lambda)\ \frac{t^n}{n!} = \lambda x \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-\lambda)^{n-k}}{k!}\ Y_k(x;\lambda)\right)t^n$$
$$-\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}Y_k(x;\lambda)2^{n-k}\left(\frac{\lambda^2}{\lambda-1}\right)^{n-k+1}\mathrm{Ch}_{n-k}\right]\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2. *Let $v \in \mathbb{N}_0$. Then*

$$\frac{\partial^v}{\partial t^v}\{F(t,x,\lambda)\} = \left[\sum_{j=0}^{v}(-1)^j (v)_j (x)_{v-j}\ \lambda^{v+j}(1+\lambda t)^{j-v}\right.$$
$$\left.\cdot\left(\lambda^2 t+\lambda-1\right)^{-j}\right]F(t,x,\lambda). \tag{17}$$

Proof. We observe that

$$\frac{\partial}{\partial t}\{F(t,x,\lambda)\} = \left[\lambda(\lambda t+1)^{-1}x - \lambda^2\left(\lambda^2 t+\lambda-1\right)^{-1}\right]F(t,x,\lambda). \tag{18}$$

Therefore, by iterating the above derivation for the variable t, the proof of Theorem 2 is completed. □

By using (1) and (17), we get

$$\frac{\partial^v}{\partial t^v}\{F(t,x,\lambda)\} = \sum_{n=0}^{\infty} Y_{n+v}(x;\lambda)\ \frac{t^n}{n!}. \tag{19}$$

We now set

$$\frac{1}{(1+\lambda t)^{v-j}} = \sum_{k=0}^{\infty}(-1)^k\binom{v-j+k-1}{k}\lambda^k t^k, \tag{20}$$

$$\left(\lambda^2 t+\lambda-1\right)^{-j} = \frac{1}{(\lambda-1)^j}\sum_{k=0}^{\infty}(-1)^k\binom{j+k-1}{k}\left(\frac{\lambda^2 t}{\lambda-1}\right)^k \tag{21}$$

and

$$\frac{1}{(1+\lambda t)^{v-j}}\frac{1}{(\lambda^2 t+\lambda-1)^j} = \frac{1}{(\lambda-1)^j}\sum_{k=0}^{\infty}C_k(j,v,\lambda)\ \frac{t^k}{k!}, \tag{22}$$

where

$$C_k(j,v,\lambda)=\sum_{m=0}^{k}(-1)^k\binom{k}{m}(v-j+k-1)_m\,(j+k-1)_{k-m}\,\frac{\lambda^{2k-m}}{(\lambda-1)^{k-m}}.$$

Substituting from (19), (20), (21) and (22) into (17) and using the Cauchy product, we obtain

$$\sum_{n=0}^{\infty}Y_{n+v}(x;\lambda)\,\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left[\sum_{j=0}^{v}(-1)^j(v)_j\,(x)_{v-j}\,\lambda^v\left(\frac{\lambda}{\lambda-1}\right)^j\cdot\sum_{k=0}^{n}\binom{n}{k}C_k(j,v,\lambda)\,Y_{n-k}(x;\lambda)\right]\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of this last equation, we arrive at the following theorem:

Theorem 3. *Let $v\in\mathbb{N}_0$. Then*

$$Y_{n+v}(x;\lambda)=\sum_{j=0}^{v}(v)_j\,(x)_{v-j}\,\lambda^v\left(-\frac{\lambda}{\lambda-1}\right)^j\cdot\sum_{k=0}^{n}\binom{n}{k}C_k(j,v,\lambda)\,Y_{n-k}(x;\lambda),\tag{23}$$

where

$$C_k(j,v,\lambda)=\sum_{m=0}^{k}(-1)^k\binom{k}{m}(v-j+k-1)_m\,(j+k-1)_{k-m}\,\frac{\lambda^{2k-m}}{(\lambda-1)^{k-m}}.$$

Now, since

$$(x)_{v-j}=\sum_{l=0}^{v-j}S_1(v-j,l)\,x^l,\tag{24}$$

upon setting $n=v-j$ in (13) and combining the resulting equation with (23), we get the following corollary.

Corollary 2. *Let $v\in\mathbb{N}_0$. Then*

$$Y_{n+v}(x;\lambda)=\sum_{j=0}^{v}(v)_j\,\lambda^v\left(-\frac{\lambda}{\lambda-1}\right)^j\sum_{l=0}^{v-j}S_1(v-j,l)\,x^l\cdot\sum_{k=0}^{n}\binom{n}{k}C_k(j,v,\lambda)\,Y_{n-k}(x;\lambda).$$

Additionally, it can be found that

$$\frac{\partial}{\partial x}\{F(t,x,\lambda)\} = F(t,x,\lambda)\log(\lambda t+1). \tag{25}$$

Thus, clearly, we have

$$\frac{1}{\log(\lambda t+1)}\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\{Y_n(x;\lambda)\}\frac{t^n}{n!} = \sum_{n=0}^{\infty}Y_n(x;\lambda)\frac{t^n}{n!}.$$

By using the equation (14), we obtain

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\frac{\partial}{\partial x}\{Y_k(x;\lambda)\}\lambda^{n-k}b_{n-k}(0)\right]\frac{t^n}{n!} = \sum_{n=0}^{\infty}\lambda n Y_{n-1}(x;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following result.

Theorem 4. *Let $n \in \mathbb{N}$. Then*

$$Y_{n-1}(x;\lambda) = \frac{1}{\lambda n}\sum_{k=0}^{n}\binom{n}{k}\frac{\partial}{\partial x}\{Y_k(x;\lambda)\}\lambda^{n-k}b_{n-k}(0).$$

3. Identities related to the PDEs for the generating function $F(t,x,\lambda)$

In this section, we give obtain partial derivative equations including the functions $F(t,x,\lambda)$; with respect to x, t and λ. By using these equations, we derive some new partial derivative formulas for the polynomials $Y_n(x;\lambda)$. We also give recurrence relations for these polynomials.

Differentiating both side of (18) with respect to x, we get the following partial differential equation:

$$\begin{aligned}\frac{\partial^2}{\partial x\partial t}\{F(t,x,\lambda)\} = \Big[&\lambda(\lambda t+1)^{-1}x\log(\lambda t+1) - \lambda^2\left(\lambda^2 t+\lambda-1\right)^{-1}\log(\lambda t+1)\\ &+\lambda(\lambda t+1)^{-1}\Big]F(t,x,\lambda).\end{aligned} \tag{26}$$

On the other hand, if we differentiate both sides of the equation (18) with respect to λ, we also get the following partial differential equation:

$$\begin{aligned}&\frac{\partial^2}{\partial\lambda\partial t}\{F(t,x,\lambda)\}\\ &\quad=\Big[\lambda t(\lambda t+1)^{-2}(x-1)x + (\lambda t+1)^{-1}x - \lambda^2 t(\lambda t+1)^{-1}\left(\lambda^2 t+\lambda-1\right)^{-1}x\end{aligned}$$

$$- \lambda(\lambda t+1)^{-1}\left(\lambda^2 t+\lambda-1\right)^{-1}(2\lambda t+1)\,x - 2\lambda\left(\lambda^2 t+\lambda-1\right)^{-1}$$
$$+ 2\lambda^2\left(\lambda^2 t+\lambda-1\right)^{-2}(2\lambda t+1)\Big]F(t,x,\lambda)\,. \tag{27}$$

If we differentiate both sides of the equation (25) with respect to λ, we also get the following partial differential equation:

$$\begin{aligned}&\frac{\partial^2}{\partial\lambda\partial x}\{F(t,x,\lambda)\}\\ &\quad= \Big[xt(\lambda t+1)^{-1}\log(\lambda t+1) - (2\lambda t+1)\left(\lambda^2 t+\lambda-1\right)^{-1}\log(\lambda t+1)\\ &\qquad + t(\lambda t+1)^{-1}\Big]F(t,x,\lambda)\,.\end{aligned} \tag{28}$$

It is time now to give our (presumably new) formulas by using the above partial differential equations.

Theorem 5. *It is asserted that*

$$\begin{aligned}&\frac{1}{(n+2)_2}\,\frac{\partial^2}{\partial\lambda\partial x}\{Y_{n+2}(x;\lambda)\}\\ &\quad= x\sum_{l=0}^{n}\sum_{k=0}^{l}(-1)^{n-l+k}\,\frac{\lambda^{n-l+k+1}}{(k+1)\,(l-k)!}\,Y_{l-k}(x;\lambda)\\ &\qquad - \lambda\sum_{l=0}^{n}\sum_{k=0}^{l}(-1)^{n-l}\binom{l}{k}\frac{\lambda^{n-l+1}}{(n-l+1)\,l!}\,Y_k(\lambda)\,Y_{l-k}(x;\lambda)\\ &\qquad - \frac{1}{2}\sum_{l=0}^{n+1}\sum_{k=0}^{l}(-1)^{n+1-l}\binom{l}{k}\frac{\lambda^{n+2-l}}{(n+2-l)\,l!}\,Y_k(\lambda)\,Y_{l-k}(x;\lambda)\\ &\qquad + \sum_{k=0}^{n+1}\frac{(-1)^k\lambda^k}{(n+1-k)!}\,Y_{n+1-k}(x;\lambda)\,.\end{aligned}$$

Proof. By using (2), we modify (28) as follows:

$$\begin{aligned}\frac{\partial^2}{\partial\lambda\partial x}\{F(t,x,\lambda)\} &= \left(\frac{xt}{\lambda t+1}\,\log(\lambda t+1) - \frac{1}{2}(2\lambda t+1)\log(\lambda t+1)\,F(t,\lambda) + \frac{t}{\lambda t+1}\right)\\ &\quad\cdot F(t,x,\lambda)\,,\end{aligned}$$

which, for $|\lambda t|<1$, readily yields

$$\frac{1}{t^2}\sum_{n=0}^{\infty}\frac{\partial^2}{\partial\lambda\partial x}\{Y_n(x;\lambda)\}\frac{t^n}{n!}=x\sum_{n=0}^{\infty}(-\lambda t)^n\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^{n+1}t^n}{n+1}\sum_{n=0}^{\infty}Y_n(x;\lambda)\ \frac{t^n}{n!}$$
$$-\left(\lambda+\frac{1}{2t}\right)\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^{n+1}t^n}{n+1}$$
$$\cdot\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}Y_k(\lambda)\,Y_{n-k}(x;\lambda)\right)\frac{t^n}{n!}$$
$$+\frac{1}{t}\sum_{n=0}^{\infty}(-\lambda t)^n\sum_{n=0}^{\infty}Y_n(x;\lambda)\ \frac{t^n}{n!}.$$

By using the Cauchy product in the above equation and after some elementary calculations, we get

$$\sum_{n=0}^{\infty}\frac{1}{(n+2)_2}\ \frac{\partial^2}{\partial\lambda\partial x}\{Y_{n+2}(x;\lambda)\}\ \frac{t^n}{n!}$$
$$=x\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}(-1)^{n-l+k}\ \frac{\lambda^{n-l+k+1}}{(k+1)\cdot(l-k)!}\ Y_{l-k}(x;\lambda)\right)t^n$$
$$-\lambda\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}(-1)^{n-l}\binom{l}{k}\frac{\lambda^{n-l+1}}{(n-l+1)\,l!}\ Y_k(\lambda)\,Y_{l-k}(x;\lambda)\right)t^n$$
$$-\frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n+1}\sum_{k=0}^{l}(-1)^{n+1-l}\binom{l}{k}\frac{\lambda^{n+2-l}}{(n+2-l)\,l!}\ Y_k(\lambda)\,Y_{l-k}(x;\lambda)\right)t^n$$
$$+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n+1}\frac{(-\lambda)^k\ (n+l)!}{(n+1-k)!}\ Y_{n+1-k}(x;\lambda)\right)\ \frac{t^n}{n!}.$$

Comparing the coefficients of t^n on both sides of this last equation, we arrive at the desired result. □

Theorem 6. *Let $n\in\mathbb{N}\setminus\{1\}$. Then the following identity holds true*:

$$2\lambda n\frac{\partial}{\partial\lambda}\{Y_n(x;\lambda)\}+2\frac{\partial}{\partial\lambda}\{Y_{n+1}(x;\lambda)\}$$
$$=2\lambda nx^2\sum_{k=0}^{n-1}\binom{n-1}{k}k!\,(-\lambda)^k\ Y_{n-1-k}(x;\lambda)+2x\sum_{k=0}^{n}\binom{n}{k}k!\ (-\lambda)^k\ Y_{n-k}(x;\lambda)$$
$$-\lambda^2(3x+2)\sum_{k=0}^{n-1}\binom{n-1}{k}nY_k(\lambda)\,Y_{n-1-k}(x;\lambda)$$
$$-\lambda(x+2)\sum_{k=0}^{n}\binom{n}{k}Y_k(\lambda)\,Y_{n-k}(x;\lambda)$$

$$+ 2\lambda^4 \sum_{l=0}^{n-2} \sum_{k=0}^{l} \binom{n-2}{l} \binom{l}{k} n\,(n-1)\, Y_{l-k}(\lambda)\, Y_k(\lambda)\, Y_{n-2-l}(x;\lambda)$$

$$+ 3\lambda^3 \sum_{l=0}^{n-1} \sum_{k=0}^{l} \binom{n-1}{l} \binom{l}{k} n Y_{l-k}(\lambda)\, Y_k(\lambda)\, Y_{n-1-l}(x;\lambda)$$

$$+ \lambda^2 \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{n}{l} \binom{l}{k} Y_{l-k}(\lambda)\, Y_k(\lambda)\, Y_{n-l}(x;\lambda)\,.$$

Proof. By using (2), we modify (27) as follows:

$$\frac{\partial^2}{\partial\lambda\partial t}\{F(t,x,\lambda)\} = \Bigg[\frac{\lambda(x-1)xt + x(\lambda t+1)}{(\lambda t+1)^2} - \left(\frac{\lambda^2 xt + \lambda x(2\lambda t+1) + 2\lambda(\lambda t+1)}{2(\lambda t+1)}\right) F(t,\lambda) + \lambda^2\left(\lambda t + \frac{1}{2}\right)[F(t,\lambda)]^2\Bigg] F(t,x,\lambda)\,.$$

Therefore, we have

$$2(\lambda t+1)\frac{\partial^2}{\partial\lambda\partial t}\{F(t,x,\lambda)\} = \Bigg[\frac{2x(\lambda xt+1)}{\lambda t+1} - \left[\lambda^2(3x+2)t + \lambda(x+2)\right] F(t,\lambda) + \left(2\lambda^4 t^2 + 3\lambda^3 t + \lambda^2\right)[F(t,\lambda)]^2\Bigg] F(t,x,\lambda)\,,$$

which readily yields

$$(2\lambda t+2)\sum_{n=0}^{\infty} \frac{\partial}{\partial\lambda}\{Y_{n+1}(x;\lambda)\}\ \frac{t^n}{n!}$$

$$= \left(2\lambda x^2 t + 2x\right)\sum_{n=0}^{\infty}(-\lambda t)^n \sum_{n=0}^{\infty} Y_n(x;\lambda)\ \frac{t^n}{n!}$$

$$- \left[\lambda^2(3x+2)t + \lambda(x+2)\right]\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k} Y_k(\lambda)\, Y_{n-k}(x;\lambda)\right]\ \frac{t^n}{n!}$$

$$+ \left(2\lambda^4 t^2 + 3\lambda^3 t + \lambda^2\right)\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}\binom{l}{k} Y_{l-k}(\lambda)\, Y_k(\lambda)\, Y_{n-l}(x;\lambda)\right]\ \frac{t^n}{n!}.$$

We thus find that

$$\sum_{n=0}^{\infty}\left(2\lambda n\frac{\partial}{\partial\lambda}\left\{Y_n\left(x;\lambda\right)\right\}+2\frac{\partial}{\partial\lambda}\left\{Y_{n+1}\left(x;\lambda\right)\right\}\right)\frac{t^n}{n!}$$

$$\begin{aligned}
&=2\lambda x^2\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n-1}\binom{n-1}{k}k!\left(-\lambda\right)^k nY_{n-1-k}\left(x;\lambda\right)\right[\frac{t^n}{n!}\\
&+2x\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}k!\left(-\lambda\right)^k Y_{n-k}\left(x;\lambda\right)\right]\frac{t^n}{n!}\\
&-\lambda^2\left(3x+2\right)\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n-1}\binom{n-1}{k}nY_k\left(\lambda\right)Y_{n-1-k}\left(x;\lambda\right)\right]\frac{t^n}{n!}\\
&-\lambda\left(x+2\right)\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}Y_k\left(\lambda\right)Y_{n-k}\left(x;\lambda\right)\right]\frac{t^n}{n!}\\
&+2\lambda^4\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n-2}\sum_{k=0}^{l}\binom{n-2}{l}\binom{l}{k}n\left(n-1\right)Y_{l-k}\left(\lambda\right)Y_k\left(\lambda\right)Y_{n-2-l}\left(x;\lambda\right)\right]\frac{t^n}{n!}\\
&+3\lambda^3\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n-1}\sum_{k=0}^{l}\binom{n-1}{l}\binom{l}{k}nY_{l-k}\left(\lambda\right)Y_k\left(\lambda\right)Y_{n-1-l}\left(x;\lambda\right)\right]\frac{t^n}{n!}\\
&+\lambda^2\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}\binom{l}{k}Y_{l-k}\left(\lambda\right)Y_k\left(\lambda\right)Y_{n-l}\left(x;\lambda\right)\right]\frac{t^n}{n!}.
\end{aligned}$$

Now, upon comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 7. *Let $n\in\mathbb{N}$. Then*

$$\begin{aligned}
\frac{\partial}{\partial x}\left\{Y_{n+1}\left(x;\lambda\right)\right\}&=x\sum_{l=0}^{n-1}\sum_{k=0}^{l}\left(-1\right)^{n-1-l}\binom{l}{k}\frac{\lambda^{n+1-l+k}n!}{l!}D_kY_{l-k}\left(x;\lambda\right)\\
&-\frac{1}{2}\sum_{l=0}^{n-1}\sum_{k=0}^{l}\binom{n-1}{l}\binom{l}{k}\lambda^{n+2-l}nD_{n-1-l}Y_k\left(\lambda\right)Y_{l-k}\left(x;\lambda\right)\\
&+\sum_{k=0}^{n}\left(-1\right)^k\frac{n!\,\lambda^{k+1}}{(n-k)!}Y_{n-k}\left(x;\lambda\right).
\end{aligned}$$

Proof. By using (2), we modify (26) as follows:

$$\begin{aligned}
\frac{\partial^2}{\partial x\partial t}\left\{F\left(t,x,\lambda\right)\right\}=&\left(\frac{\lambda x}{\lambda t+1}\log\left(\lambda t+1\right)-\frac{\lambda^2}{2}\log\left(\lambda t+1\right)F\left(t,\lambda\right)+\frac{\lambda}{\lambda t+1}\right)\\
&\cdot F\left(t,x,\lambda\right).
\end{aligned}\tag{29}$$

Upon rearranging right-hand side of the equation (29), if we make use of the generating function for the Daehee numbers in (5), we get the following functional equation:

$$\frac{\partial^2}{\partial x \partial t}\{F(t,x,\lambda)\} = \left(\frac{\lambda^2 xt}{\lambda t+1}F_D(\lambda t) - \frac{\lambda^3 t}{2}F_D(\lambda t)F(t,\lambda) + \frac{\lambda}{\lambda t+1}\right)F(t,x,\lambda),$$

which can be used to obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\{Y_{n+1}(x;\lambda)\}\frac{t^n}{n!} = \Bigg(&\lambda^2 xt\sum_{n=0}^{\infty}(-\lambda t)^n\sum_{n=0}^{\infty}D_n\frac{(\lambda t)^n}{n!}\\ &-\frac{\lambda^3 t}{2}\sum_{n=0}^{\infty}D_n\frac{(\lambda t)^n}{n!}\sum_{n=0}^{\infty}Y_n(\lambda)\frac{t^n}{n!}\\ &+\lambda\sum_{n=0}^{\infty}(-\lambda t)^n\Bigg)\sum_{n=0}^{\infty}Y_n(x;\lambda)\frac{t^n}{n!}.\end{aligned}$$

By using the Cauchy product, we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\{Y_{n+1}(x;\lambda)\}\frac{t^n}{n!}\\ &= \lambda^2 xt\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\sum_{k=0}^{l}(-1)^{n-l}\binom{l}{k}\frac{\lambda^{n-l+k}\,n!}{l!}D_kY_{l-k}(x;\lambda)\right]\frac{t^n}{n!}\\ &\quad-\frac{\lambda^3 t}{2}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}\binom{l}{k}\lambda^{n-l}D_{n-l}Y_k(\lambda)Y_{l-k}(x;\lambda)\frac{t^n}{n!}\\ &\quad+\lambda\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k\frac{\lambda^k\,n!}{(n-k)!}Y_{n-k}(x;\lambda)\right)\frac{t^n}{n!}.\end{aligned}$$

Hence we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\{Y_{n+1}(x;\lambda)\}\frac{t^n}{n!}\\ &= \lambda^2 x\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n-1}\sum_{k=0}^{l}(-1)^{n-1-l}\binom{l}{k}\frac{\lambda^{n-1-l+k}\,n!}{l!}D_kY_{l-k}(x;\lambda)\right]\frac{t^n}{n!}\\ &\quad-\frac{\lambda^3}{2}\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n-1}\sum_{k=0}^{l}\binom{n-1}{l}\binom{l}{k}\lambda^{n-1-l}nD_{n-1-l}Y_k(\lambda)Y_{l-k}(x;\lambda)\right]\frac{t^n}{n!}\\ &\quad+\lambda\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{n!(-\lambda)^k}{(n-k)!}Y_{n-k}(x;\lambda)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Lemma 1. *It is asserted that*

$$Y_n(x;-1) = (-1)^{n+1}\,\mathrm{Ch}_n(x)\,. \tag{30}$$

Proof. Upon setting $\lambda = -1$ in (1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\,Y_n(x;-1)\,\frac{t^n}{n!} &= \frac{2(1-t)^x}{t-2}\\ &= -\sum_{n=0}^{\infty}\mathrm{Ch}_n(x)\,\frac{t^n}{n!}\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Lemma 2. *The following relationship holds true*:

$$Y_n(-1) = (-1)^{n+1}\,\mathrm{Ch}_n. \tag{31}$$

Proof. Substituting $x = 0$ into (30), we arrive at the desired result. □

Corollary 3. *The following derivative formula holds true*:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\{\mathrm{Ch}_{n+1}(x)\} = x\sum_{l=0}^{n-1}\sum_{k=0}^{l}(-1)^{n+l+1}\binom{l}{k}\frac{n!}{l!}\,D_k\mathrm{Ch}_{l-k}(x)\\ -\frac{1}{2}\sum_{l=0}^{n-1}\sum_{k=0}^{l}\binom{n-1}{l}\binom{l}{k}nD_{n-1-l}\mathrm{Ch}_k\mathrm{Ch}_{l-k}(x)\\ +\sum_{k=0}^{n}\frac{(-1)^k\,n!}{(n-k)!}\,\mathrm{Ch}_{n-k}(x)\,.\end{aligned}$$

Proof. By combining (31) and (30) with Theorem 7, we arrive at the desired result. □

Theorem 8. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}\lambda\,\frac{\partial}{\partial x}\{Y_{n+1}(x;\lambda)\}+\frac{1}{n+1}\,\frac{\partial}{\partial x}\{Y_{n+2}(x;\lambda)\}\\ = x\sum_{k=0}^{n}\frac{(-1)^k\,\lambda^{k+2}\,n!}{(k+1)(n-k)!}\,Y_{n-k}(x;\lambda)\end{aligned}$$

$$-\frac{1}{2}\sum_{l=0}^{n-1}\sum_{k=0}^{l}\frac{(-1)^{n-1-l}\lambda^{n+3-l}}{n-l}\binom{l}{k}\frac{n!}{l!}Y_k(\lambda)Y_{l-k}(x;\lambda)$$

$$-\frac{1}{2}\sum_{l=0}^{n}\frac{(-1)^{n-l}\lambda^{n+3-l}}{n-l+1}\sum_{k=0}^{l}\binom{l}{k}\frac{n!}{l!}Y_k(\lambda)Y_{l-k}(x;\lambda)$$

$$+\frac{\lambda}{n+1}Y_{n+1}(x;\lambda).$$

Proof. Multiplying both sides of (29) by $\lambda t+1$, we have

$$(\lambda t+1)\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\{Y_{n+1}(x;\lambda)\}\frac{t^n}{n!}=\lambda x\sum_{n=0}^{\infty}\frac{(-1)^n(\lambda t)^{n+1}}{n+1}\sum_{n=0}^{\infty}Y_n(x;\lambda)\frac{t^n}{n!}$$
$$-\frac{\lambda^3 t+\lambda^2}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(\lambda t)^{n+1}}{n+1}$$
$$\cdot\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}Y_k(\lambda)Y_{n-k}(x;\lambda)\right)\frac{t^n}{n!}$$
$$+\lambda\sum_{n=0}^{\infty}Y_n(x;\lambda)\frac{t^n}{n!}.$$

Multiplying both sides of this last equation by $\frac{1}{t}$ and by using the Cauchy product in the resulting equation, we get

$$\lambda\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\{Y_{n+1}(x;\lambda)\}\frac{t^n}{n!}+\frac{1}{t}\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\{Y_{n+1}(x;\lambda)\}\frac{t^n}{n!}$$
$$=\lambda x\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^k\lambda^{k+1}}{(k+1)(n-k)!}Y_{n-k}(x;\lambda)\right)t^n$$
$$-\frac{\lambda^3 t}{2}\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\frac{(-1)^{n-l}\lambda^{n-l+1}}{(n-l+1)\cdot l!}\sum_{k=0}^{l}\binom{l}{k}Y_k(\lambda)Y_{l-k}(x;\lambda)\right]t^n$$
$$-\frac{\lambda^2}{2}\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\frac{(-1)^{n-l}\lambda^{n-l+1}}{(n-l+1)\cdot l!}\sum_{k=0}^{l}\binom{l}{k}Y_k(\lambda)Y_{l-k}(x;\lambda)\right]t^n$$
$$+\frac{\lambda}{t}\sum_{n=0}^{\infty}Y_n(x;\lambda)\frac{t^n}{n!}.$$

Hence we have

$$\lambda \sum_{n=0}^{\infty} \frac{\partial}{\partial x}\{Y_{n+1}(x;\lambda)\} \frac{t^n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{\partial}{\partial x}\{Y_{n+2}(x;\lambda)\} \frac{t^n}{n!}$$

$$= \lambda x \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^k \lambda^{k+1} n!}{(k+1)(n-k)!} Y_{n-k}(x;\lambda) \right) \frac{t^n}{n!}$$

$$- \frac{\lambda^3}{2} \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n-1} \sum_{k=0}^{l} \frac{(-1)^{n-1-l} \lambda^{n-l}}{n-l} \binom{l}{k} \frac{n!}{l!} Y_k(\lambda) Y_{l-k}(x;\lambda) \right) \frac{t^n}{n!}$$

$$- \frac{\lambda^2}{2} \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \frac{(-1)^{n-l} \lambda^{n-l+1}}{n-l+1} \sum_{k=0}^{l} \binom{l}{k} \frac{n!}{l!} Y_k(\lambda) Y_{l-k}(x;\lambda) \right) \frac{t^n}{n!}$$

$$+ \lambda \sum_{n=0}^{\infty} \frac{1}{n+1} Y_{n+1}(x;\lambda) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

4. Identities and relations associated with $Y_n(\lambda)$ and $Y_n(x;\lambda)$

In this section, we give further identities and relations related to the numbers $Y_n(\lambda)$ including the Hurwitz–Lerch zeta functions, the Apostol–Bernoulli numbers, the Apostol–Euler numbers. Furthermore, we give some hypergeometric function representations for the integrals of the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$. Finally, we give some infinite series representation including the numbers $Y_n(\lambda)$, the Daehee numbers, the Changhee numbers, the Lucas numbers, and the Humbert polynomials.

4.1. Identities for the numbers $Y_n(\lambda)$ involving the Apostol–Bernoulli numbers and the Apostol–Euler numbers

Here, in this subsection, we give several identities related to the numbers $Y_n(\lambda)$ including not only the Apostol–Bernoulli numbers and their interpolation function (that is, the Hurwitz–Lerch zeta function), but also the Apostol–Euler numbers.

Theorem 9. *Let $k \in \mathbb{N}_0$. Then*

$$Y_k(\lambda) = 2\lambda^k \sum_{m=0}^{k} \frac{S_1(k,m)\,\mathcal{B}_{m+1}(\lambda)}{m+1}.$$

Proof. Replacing $1+\lambda t$ by $e^{\log(1+\lambda t)}$ in (2), for $\left|\lambda e^{\log(1+\lambda t)}\right| < 1$, we have

$$\sum_{k=0}^{\infty} Y_k(\lambda) \frac{t^k}{k!} = \frac{2}{\lambda e^{\log(1+\lambda t)} - 1}$$

$$= -2\sum_{n=0}^{\infty} \lambda^n \, e^{n\log(1+\lambda t)}$$

$$= -2\sum_{n=0}^{\infty} \lambda^n \sum_{m=0}^{\infty} \frac{[n\log(1+\lambda t)]^m}{m!}$$

$$= -2\sum_{n=0}^{\infty} \lambda^n \sum_{m=0}^{\infty} n^m \sum_{k=0}^{\infty} S_1(k,m)\, \frac{(\lambda t)^k}{k!}$$

$$= -2\sum_{k=0}^{\infty} \left(\sum_{m=0}^{\infty} S_1(k,m) \sum_{n=0}^{\infty} \lambda^n n^m\right) \frac{(\lambda t)^k}{k!}.$$

Comparing the coefficients of $\frac{t^k}{k!}$ on both sides of the above equation, we obtain

$$Y_k(\lambda) = -2\lambda^k \sum_{m=0}^{\infty} S_1(k,m) \sum_{n=0}^{\infty} \lambda^n n^m. \tag{32}$$

It is well known that the Apostol–Bernoulli numbers are interpolated by the Hurwitz–Lerch zeta function (*cf.* [1,3,4,34–36]) with the following relation:

$$\Phi(\lambda, -m, 0) = \sum_{n=0}^{\infty} \lambda^n n^m = -\frac{\mathcal{B}_{m+1}(\lambda)}{m+1}.$$

By using this last identity in (32), we arrive at the desired result. □

Theorem 10. *Let* $m \in \mathbb{N}_0$. *Then*

$$Y_m(-\lambda) = (-1)^{m+1} \lambda^m \sum_{n=0}^{m} \mathcal{E}_n(\lambda)\, S_1(m,n). \tag{33}$$

Proof. We observe that

$$\sum_{m=0}^{\infty} Y_m(-\lambda)\, \frac{t^m}{m!} = \frac{2}{-\left(\lambda e^{\log(1-\lambda t)} + 1\right)}$$

$$= -\sum_{n=0}^{\infty} \mathcal{E}_n(\lambda)\, \frac{[\log(1-\lambda t)]^n}{n!}$$

$$= -\sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} (-\lambda)^m \mathcal{E}_n(\lambda)\, S_1(m,n)\right) \frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the desired result. □

Remark 1. By setting $\lambda = 1$ into the equation (33), we have

$$Y_m(-1) = (-1)^{m+1} \sum_{n=0}^{m} \mathcal{E}_n(1) S_1(m,n)$$

and combining the resulting equation with (31) and (12), we obtain

$$\mathrm{Ch}_m = \sum_{n=0}^{m} E_n S_1(m,n),$$

which was proven by Kim et al. [15, Theorem 2.7].

Theorem 11. *Let* $m \in \mathbb{N}_0$. *Then*

$$Y_m(x;-\lambda) = (-1)^{m+1} \lambda^m \sum_{n=0}^{m} \mathcal{E}_n(x;\lambda) S_1(m,n). \tag{34}$$

Proof. It is observed that

$$\begin{aligned}\sum_{m=0}^{\infty} Y_m(x;-\lambda) \frac{t^m}{m!} &= \frac{2e^{x\log(1-\lambda t)}}{-\left(\lambda e^{\log(1-\lambda t)}+1\right)} \\ &= -\sum_{n=0}^{\infty} \mathcal{E}_n(x;\lambda) \frac{[\log(1-\lambda t)]^n}{n!} \\ &= -\sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} (-\lambda)^m \mathcal{E}_n(\lambda) S_1(m,n)\right) \frac{t^m}{m!}\end{aligned}$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the desired result. □

Remark 2. By setting $\lambda = 1$ in the equation (34), we have

$$Y_m(x;-1) = (-1)^{m+1} \sum_{n=0}^{m} \mathcal{E}_n(x;1) S_1(m,n)$$

and combining the resulting equation with (30) and (11), we find that

$$\mathrm{Ch}_m(x) = \sum_{n=0}^{m} E_n(x) S_1(m,n),$$

which was proven by Kim et al. [15, Theorem 2.5].

4.2. Hypergeometric function representations for the integrals of the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$

Here, in this subsection, we give hypergeometric function representations for the integrals of not only the numbers $Y_n(\lambda)$ which is a function of the parameter λ, but also the polynomials $Y_n(x;\lambda)$.

Theorem 12. *It is asserted that*

$$\int_0^u Y_n(\lambda)\,\mathrm{d}\lambda = \frac{-2\cdot n!u^{2n+1}}{2n+1}\,{}_2F_1\left(-n-1,-2n-1;-2n-2;-u\right),$$

where ${}_2F_1$ denotes the Gauss hypergeometric functions.

Proof. We know that the following explicit formula holds true for the numbers $Y_n(\lambda)$ (see [30, p. 15, Theorem 14]):

$$Y_n(\lambda) = (-1)^n\,\frac{2\cdot n!}{\lambda-1}\left(\frac{\lambda^2}{\lambda-1}\right)^n.$$

Integrating both sides of the above equation with respect to λ, we have

$$\begin{aligned}\int_0^u Y_n(\lambda)\,\mathrm{d}\lambda &= \int_0^u \frac{2\cdot n!}{\lambda-1}\left(-\frac{\lambda^2}{\lambda-1}\right)^n \mathrm{d}\lambda \\ &= -2\cdot n!\int_0^u \frac{\lambda^{2n}}{(1-\lambda)^{n+1}}\,\mathrm{d}\lambda \\ &= -2\cdot n!\int_0^u \sum_{k=0}^{\infty}\binom{n+k}{k}\lambda^{2n+k}\,\mathrm{d}\lambda.\end{aligned}$$

We thus find that

$$\int_0^u Y_n(\lambda)\,\mathrm{d}\lambda = -2u^{2n+1}\sum_{k=0}^{\infty}\frac{(n+k)!}{2n+k+1}\,\frac{u^k}{k!}. \tag{35}$$

Therefore, we get

$$\int_0^u Y_n(\lambda)\,\mathrm{d}\lambda = -2\cdot n!\,u^{2n+1}\sum_{k=0}^{\infty}\frac{(n+1)^{(k)}\,(2n+1)^{(k)}}{(2n+2)^{(k)}\,(2n+1)}\,\frac{u^k}{k!}.$$

By using (3), the equation (35) reduces to the following integral:

$$\int_0^u Y_n(\lambda)\,\mathrm{d}\lambda = \frac{-2\cdot n!\,u^{2n+1}}{2n+1}\sum_{k=0}^{\infty}\frac{(-n-1)_k\,(-2n-1)_k}{(-2n-2)_k}\,\frac{(-u)^k}{k!}.$$

Hence we arrive at the desired result. □

Theorem 13. *It is asserted that*

$$\int_0^u Y_n(x;\lambda)\,\mathrm{d}\lambda = -2\cdot n! u^{2n+1}\sum_{k=0}^{n}\binom{x}{k}\frac{u^{-k}}{2n-k+1}$$
$$\cdot\ {}_2F_1\left(k-n-1,k-2n-1;k-2n-2;-u\right).$$

Proof. If, in the equation (1), we assume that

$$\left|\lambda^2 t\right| < |\lambda-1|$$

and use the Cauchy product, we get

$$\sum_{n=0}^{\infty} Y_n(x;\lambda)\frac{t^n}{n!} = \frac{2}{\lambda-1}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{n-k}\binom{x}{k}\lambda^k\left(\frac{\lambda^2}{\lambda-1}\right)^{n-k}\right)t^n.$$

We thus obtain

$$Y_n(x;\lambda) = \frac{2\cdot n!}{\lambda-1}\left(\frac{\lambda^2}{\lambda-1}\right)^n\sum_{k=0}^{n}(-1)^{n-k}\binom{x}{k}\left(\frac{\lambda-1}{\lambda}\right)^k.$$

Integrating both sides of the above equation with respect to λ, we get

$$\begin{aligned}
\int_0^u Y_n(x;\lambda)\mathrm{d}\lambda &= \int_0^u \frac{2\cdot n!}{\lambda-1}\left(\frac{\lambda^2}{\lambda-1}\right)^n\sum_{k=0}^{n}(-1)^{n-k}\binom{x}{k}\left(\frac{\lambda-1}{\lambda}\right)^k\,\mathrm{d}\lambda\\
&= 2\cdot n!\sum_{k=0}^{n}(-1)^{n-k}\binom{x}{k}\int_0^u\frac{\lambda^{2n-k}}{(\lambda-1)^{n-k+1}}\,\mathrm{d}\lambda\\
&= -2\cdot n!\sum_{k=0}^{n}\binom{x}{k}\int_0^u\frac{\lambda^{2n-k}}{(1-\lambda)^{n-k+1}}\,\mathrm{d}\lambda\\
&= -2\cdot n!\sum_{k=0}^{n}\binom{x}{k}\int_0^u\lambda^{2n-k}\sum_{m=0}^{\infty}\binom{n-k+m}{m}\lambda^m\,\mathrm{d}\lambda\\
&= -2\cdot n!\sum_{k=0}^{n}\binom{x}{k}\sum_{m=0}^{\infty}\binom{n-k+m}{m}\int_0^u\lambda^{2n-k+m}\,\mathrm{d}\lambda
\end{aligned}$$

$$= -2 \cdot n! \sum_{k=0}^{n} \binom{x}{k} \sum_{m=0}^{\infty} \binom{n-k+m}{m} \frac{u^{2n-k+m+1}}{2n-k+m+1}.$$

We thus obtain

$$\int Y_n(x;\lambda)\, \mathrm{d}\lambda = -2 \cdot n! \lambda^{2n+1} \sum_{k=0}^{n} \binom{x}{k} \lambda^{-k} \sum_{m=0}^{\infty} \frac{(n-k+m)_m}{2n-k+m+1} \frac{\lambda^m}{m!}$$
$$= -2 \cdot n! \lambda^{2n+1} \sum_{k=0}^{n} \binom{x}{k} \lambda^{-k} \sum_{m=0}^{\infty} \frac{(n-k+1)^{(m)}\,(2n-k+1)^{(m)}}{(2n-k+2)^{(m)}\,(2n-k+1)} \frac{\lambda^m}{m!},$$

which, by using (3), reduces to the following equation:

$$\int_0^u Y_n(x;\lambda)\, \mathrm{d}\lambda = -2 \cdot n!\, u^{2n+1} \sum_{k=0}^{n} \binom{x}{k} \frac{u^{-k}}{2n-k+1}$$
$$\cdot \sum_{m=0}^{\infty} \frac{(k-n-1)_m\,(k-2n-1)_m}{(k-2n-2)_m} \frac{(-u)^m}{m!}.$$

Hence we arrive at the desired result. □

4.3. Infinite series representations involving the numbers $Y_n(\lambda)$

In this subsection, we give some infinite series representations involving the numbers $Y_n(\lambda)$, their inverses and the Changhee numbers, the Daehee numbers and the Lucas numbers.

Let us begin with the following series including inverses of the numbers $Y_n(\lambda)$:

$$\sum_{n=0}^{\infty} \frac{1}{Y_n(\lambda)} = \frac{\lambda-1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda-1}{\lambda^2}\right)^n.$$

We thus arrive at the following result.

Theorem 14. *It is asserted that*

$$\sum_{n=0}^{\infty} \frac{1}{Y_n(\lambda)} = \frac{\lambda-1}{2} \exp\left(-\frac{\lambda-1}{\lambda^2}\right).$$

We now proceed to give series including quotients of the Daehee numbers and the numbers $Y_n(\lambda)$. For this purpose in view, we consider

$$\sum_{n=0}^{\infty} \frac{Y_n(\lambda)}{D_n} = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2 \cdot n!}{\lambda-1} \left(\frac{\lambda^2}{\lambda-1}\right)^n \frac{n+1}{(-1)^n\, n!}$$

$$= \frac{2}{\lambda - 1} \sum_{n=0}^{\infty} (n+1) \left(\frac{\lambda^2}{\lambda - 1} \right)^n .$$

Now, assuming that

$$\left| \frac{\lambda^2}{\lambda - 1} \right| < 1,$$

we arrive at the following result.

Theorem 15. *Let*

$$\left| \frac{\lambda^2}{\lambda - 1} \right| < 1.$$

Then

$$\sum_{n=0}^{\infty} \frac{Y_n(\lambda)}{D_n} = \frac{2\lambda^2}{(1 - \lambda + \lambda^2)^2} - \frac{2}{1 - \lambda + \lambda^2}. \tag{36}$$

On the other hand, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{D_n}{Y_n(\lambda)} &= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n+1} \frac{\lambda - 1}{(-1)^n 2 \cdot n!} \left(\frac{\lambda - 1}{\lambda^2} \right)^n \\ &= \frac{\lambda - 1}{2} \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\frac{\lambda - 1}{\lambda^2} \right)^n \\ &= -\frac{\lambda^2}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{1 - \lambda}{\lambda^2} \right)^{n+1}, \end{aligned}$$

which leads us to the following result.

Theorem 16. *The following sum holds true*:

$$\sum_{n=0}^{\infty} \frac{D_n}{Y_n(\lambda)} = -\frac{\lambda^2}{2} \log \left(1 + \frac{1 - \lambda}{\lambda^2} \right).$$

Remark 3. In the special case when $\lambda = -1$, Theorem 16 yields

$$\sum_{n=0}^{\infty} \frac{D_n}{Y_n(-1)} = -\frac{\log 2}{2}.$$

Remark 4. Since

$$\sum_{n=1}^{\infty} \frac{L_n}{n \cdot 2^n} = 2 \log 2,$$

where L_n are the Lucas numbers (*cf.* [22, p. 7]), we find from Remark 3 that

$$\sum_{n=1}^{\infty}\left(\frac{D_n}{Y_n(-1)}+\frac{L_n}{n\cdot 2^{n+2}}\right)=1.$$

Thirdly, let us give series including quotients of the Changhee numbers and the numbers $Y_n(\lambda)$. We begin by observing that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{Y_n(\lambda)}{\mathrm{Ch}_n}&=\sum_{n=0}^{\infty}\frac{(-1)^n\,2\cdot n!}{\lambda-1}\left(\frac{\lambda^2}{\lambda-1}\right)^n\frac{2^n}{(-1)^n\,n!}\\&=\frac{2}{\lambda-1}\sum_{n=0}^{\infty}2^n\left(\frac{\lambda^2}{\lambda-1}\right)^n.\end{aligned}$$

Assuming that

$$\left|\frac{\lambda^2}{\lambda-1}\right|<\frac{1}{2},$$

we thus obtain the following result.

Theorem 17. *Let*

$$\left|\frac{\lambda^2}{\lambda-1}\right|<\frac{1}{2}.$$

Then

$$\sum_{n=0}^{\infty}\frac{Y_n(\lambda)}{\mathrm{Ch}_n}=\frac{2}{\lambda-1-2\lambda^2}. \tag{37}$$

Finally, we establish the identities given by Theorem 18 below.

Theorem 18. *Let*

$$\left|\frac{\lambda-1}{2\lambda^2}\right|<1.$$

Then each of the following sums holds true:

$$\sum_{n=0}^{\infty}\frac{\mathrm{Ch}_n}{Y_n(\lambda)}=\frac{\lambda^2(\lambda-1)}{2\lambda^2-\lambda+1} \tag{38}$$

and

$$\sum_{n=0}^{\infty}\frac{\mathrm{Ch}_n}{D_n}=4. \tag{39}$$

Proof. If we consider

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\mathrm{Ch}_n}{Y_n(\lambda)} &= \sum_{n=0}^{\infty}\frac{(-1)^n n!}{2^n}\frac{\lambda-1}{2(-1)^n\ n!}\left(\frac{\lambda-1}{\lambda^2}\right)^n \\ &= \frac{\lambda-1}{2}\sum_{n=0}^{\infty}\left(\frac{\lambda-1}{2\lambda^2}\right)^n,\end{aligned}$$

then we get the assertion (38) of Theorem 18.

Next, in order to derive infinite series in (39) involving the quotient of the Changhee numbers and the Daehee numbers, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\mathrm{Ch}_n}{D_n} &= \sum_{n=0}^{\infty}\frac{(-1)^n\ n!}{2^n}\frac{n+1}{(-1)^n\ n!} \\ &= 2\sum_{n=0}^{\infty}\frac{n+1}{2^{n+1}},\end{aligned}$$

which is precisely the assertion (39) of Theorem 18. □

4.4. Further remarks concerning the numbers $Y_n(\lambda)$ and the Humbert polynomials

In this subsection, we give various relations between infinite series obtained in the previous section and the Humbert polynomials.

Remark 5. By rewriting the right-hand side of the equation (36) in terms of the Humbert polynomials, we find that

$$\sum_{n=0}^{\infty}\frac{Y_n(\lambda)}{D_n} = 2\lambda^2\sum_{n=0}^{\infty}\Pi_{n,2}^{(2)}\left(\frac{1}{2}\right)\lambda^n - 2\sum_{n=0}^{\infty}\Pi_{n,2}^{(1)}\left(\frac{1}{2}\right)\lambda^n. \tag{40}$$

Remark 6. By rewriting the right-hand side of the equations (37) and (38) in terms of the generalized Humbert polynomials, we are led to the following results:

$$\sum_{n=0}^{\infty}\frac{Y_n(\lambda)}{\mathrm{Ch}_n} = -2\sum_{n=0}^{\infty}P_n\left(2,\frac{1}{2},2,-1,1\right)\lambda^n \tag{41}$$

and

$$\sum_{n=0}^{\infty}\frac{\mathrm{Ch}_n}{Y_n(\lambda)} = \lambda^2(\lambda-1)\sum_{n=0}^{\infty}P_n\left(2,\frac{1}{2},2,-1,1\right)\lambda^n. \tag{42}$$

5. Concluding remarks and observations

In the present paper, we have investigated some derivative properties of the generating functions for the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$, which were recently introduced by Simsek [30]. We have given functional equations and differential equations (PDEs) of these generating functions. By using these functional and differential equations, we have derived not only recurrence relations, but also several other identities and relations for these numbers and polynomials. Our identities include the Apostol–Bernoulli numbers, the Apostol–Euler numbers, the Stirling numbers of the first kind, the Cauchy numbers and the Hurwitz–Lerch zeta function. Moreover, we have given hypergeometric function representation for an integral involving some of these numbers and polynomials. Finally, we have derived infinite series representations of the numbers $Y_n(\lambda)$, the Changhee numbers, the Daehee numbers, the Lucas numbers and the Humbert polynomials. We have also considered various (known or new) special cases and consequences of some of the main results which we have derived in this paper.

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